University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

Problems 10-15 give you an opportunity for practice in working with spacetime four-vectors and Lorentz transformations:

10. (Light cone)
(a.)
Event A occurs at spacetime point $(ct, x, y, z) = (0, 1, 1, 1)$; event B occurs at $(1, 0, 0, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events A and B occur at the same spatial coordinates? If so, find $c|t'_A - t'_B|$, c times the magnitude of the time interval in $\mathcal{S}'$ between the two events.
(b.)
Is there an inertial system $\mathcal{S}''$ in which events A and B occur simultaneously? If so, find $|\vec{r}''_A - \vec{r}''_B|$, the distance in $\mathcal{S}''$ between the two events.
(c.)
Can event A be the cause of event B, or vice versa? Explain.
(d.)
Event D occurs at spacetime point $(ct, x, y, z) = (-1, 0, 0, 0)$; event E occurs at $(2, 1, 1, 0)$, both in an inertial system $\mathcal{S}$. Is there an inertial system $\mathcal{S}'$ in which events D and E occur simultaneously? If so, find $|\vec{r}'_E - \vec{r}'_D|$, the magnitude of the distance in $\mathcal{S}'$ between the two events.
(e.)
Is there an inertial system $\mathcal{S}''$ in which events D and E occur at the same spatial coordinates? If so, find $c|t''_E - t''_D|$, c times the magnitude of the time interval in $\mathcal{S}''$ between the two events.

11.
Using *e.g.* the method of Short Course in Special Relativity [SCSR] §7, obtain the 4×4 Lorentz transformation matrix for the case in which frame $\mathcal{S}'$ moves with respect to frame $\mathcal{S}$ with speed $\beta_0 c$ in an *arbitrary* direction $(n_1, n_2, 0)$ in the x-y plane, where $\vec{n}$ is a unit vector.

12.
(a.)
In SCSR §8, clock time intervals measured in a frame in which the clock is not at rest are shown to be *dilated* by the factor γ_0. This analysis used the *inverse* Lorentz transformation. Reanalyze the same problem using the *direct* Lorentz transformation. Is the answer the same?
(b.)
In SCSR §9, the length of a rod measured in a frame in which the rod is not at rest is shown to be *contracted* by the factor $1/\gamma_0$. This analysis used the *direct* Lorentz transformation. Reanalyze the same problem using the *inverse* Lorentz transformation. Is the answer the same?

13. (Addition of velocities)
In texts that do not emphasize the rapidity or boost parameter η, the Einstein law for the addition of velocities is derived less elegantly as follows (see SCSR Fig. 7). Denote by x^1 (x'^1) the x coordinate of the origin of $\mathcal{S}''$ as observed in the lab frame $\mathcal{S}$ (moving frame $\mathcal{S}'$). Write a standard inverse Lorentz transformation

$$\begin{aligned} x^0 &= \gamma x'^0 + \gamma\beta x'^1 \\ x^1 &= \gamma x'^1 + \gamma\beta x'^0 \ . \end{aligned}$$

Then take the differential of it: $dx^0 = \ldots$; $dx^1 = \ldots$. Divide the bottom by the top equation and identify

$$\begin{aligned} \frac{dx^1}{dx^0} &= \beta'' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S} \\ \frac{dx'^1}{dx'^0} &= \beta' = c^{-1} \times \text{ speed of } \mathcal{S}'' \text{ in } \mathcal{S}' \ . \end{aligned}$$

Obtain the Einstein law for the addition of velocities (SCSR Eq. (24)):

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \ .$$

14.
Consider the standard case in which two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$ coincide at $t = t' = 0$, with frame $\mathcal{S}'$ moving at velocity $\beta c\,\hat{x}$ with respect to

frame $\mathcal{S}$. As seen in a third frame $\mathcal{S}''$, also moving along $\hat{x}$ with respect to $\mathcal{S}$, two clocks fixed to the origins of frames $\mathcal{S}$ and $\mathcal{S}'$, respectively, appear to agree. With respect to frame $\mathcal{S}$, considering that *rapidity ("boost") is the additive parameter of the Lorentz transformation*, show that the speed $\beta''c$ of frame $\mathcal{S}''$ is given by

$$\beta'' = \tanh\left(\tfrac{1}{2}\tanh^{-1}\beta\right) .$$

15. (Taylor & Wheeler problem 51)
The clock paradox, version 3.
Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)
(a.)
The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year = 31.6×10^6 seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is* g.
(b.)
How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity* βc is not the simple quantity to analyze. The simple quantity is the *boost parameter* η. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = c\,\tanh(d\eta) \approx c\,d\eta$ so that

$$c\,d\eta = g\,d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c}d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.
(c.)
What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c\tanh\eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c\,\tanh\eta\,dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh\eta\,d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c\,\tanh\eta\,\cosh\eta\,d\tau = c\,\sinh\eta\,d\tau$$

Use the expression $c\eta = g\tau$ from part (b.) to obtain

$$dx = c \sinh\left(\frac{g\tau}{c}\right) d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.
(d.)
Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.